

# SUPERSYMMETRY OF THE SCHRÖDINGER AND KORTEWEG–DE VRIES OPERATORS

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**ABSTRACT.** In 70's A.A. Kirillov interpreted the (stationary) *Schrödinger* or *Sturm-Liouville* operator  $L_2 = \frac{d^2}{dx^2} + F$  as an element of the dual space  $\hat{\mathfrak{g}}^*$  to  $\hat{\mathfrak{g}} = \mathfrak{vir}$ , the Virasoro algebra; he also interpreted the (stationary) *KdV operator*  $L_3 = \frac{d^3}{dx^3} + \frac{d}{dx}F + F\frac{d}{dx}$  in terms of the stabilizer of  $L_2$  and found a supersymmetry that connects solutions of  $L_3f = 0$  with solutions of  $L_2g = 0$ .

We extend Kirillov's results and find all supersymmetric extension of the Schrödinger and Korteweg–de Vries operators.

We also superize the construction Khesin–Malikov's generalization of Drinfeld–Sokolov's reduction to the pseudodifferential operators.

To Alexandr Alexandrovich Kirillov  
who taught one of us

## INTRODUCTION

In 70's A.A. Kirillov made several amazing observations. In [Ki1] he associated the (stationary) *KdV operator*  $L_3 = \frac{d^3}{dx^3} + \frac{d}{dx}F + F\frac{d}{dx}$  and the (stationary) *Schrödinger* or *Sturm-Liouville* operator  $L_2 = \frac{d^2}{dx^2} + F$  with the cocycle that determines the nontrivial central extension  $\hat{\mathfrak{g}} = \mathfrak{vir}$  — the Virasoro algebra — of the Witt algebra  $\mathfrak{g} = \mathfrak{witt} = \mathfrak{der}\mathbb{C}[x^{-1}, x]$ . Moreover, he found an explanation of the commonly known fact that the product of two solutions  $f_1, f_2$  of the Schrödinger equation  $L_2(f) = 0$  satisfies  $L_3(f_1f_2) = 0$ . Kirillov's explanation: a supersymmetry [Ki2].

Kirillov also classified the orbits of the coadjoint representation of  $\mathfrak{vir}$  and clarified its equivalence to the following important classification problems: of symplectic leaves of the second Gelfand–Dickey structure on the 2nd order differential operators, of projective structures on the circle and of Hill equation (i.e., the Schrödinger equation with periodic potential).

Kirillov's approach clarifies the earlier results by Poincaré, N. Kuiper, Lazutkin and Pankratova. The recent announcement of the classification of the simple stringy superalgebras [GLS] describes, clearly, the scope of the problem: there are exactly 14 ways to superize the above result of Kirillov. (Kirillov himself partly considered one of them; several first few of the 14 possibilities were considered by V. Ovsienko and O. Ovsienko, Radul, Khesin and others, see [L2], [KM], [BIK], [DI], [DIK].)

We extend Kirillov's results and find all supersymmetric extension of the Schrödinger and Korteweg–de Vries operators associated with the 10 distinguished stringy superalgebras, i.e., all the simple stringy superalgebras that possess a nontrivial central extension.

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We also superize the construction due to Khesin–Malikov (Drinfeld–Sokolov’s reduction for the pseudodifferential operators) and relate the complex powers of the Schrödinger operators we describe to the superized KdV-type hierarchies labelled by complex parameter.

Our construction brings the KdV-type equations directly in the Lax form guaranteeing their complete integrability. The examples of  $N$ -extended KdV-type equations due to Chaichian–Kulish, Kupersmidt, P. Mathieu, Ivanov–Krivonos–Bellucci–Delduc and others are only connected with several of the distinguished stringy superalgebras and not all of the cocycles we consider.

Our superizations of KdV are completely integrable by construction. On the other hand, physicists can construct by a different method *parametric* families of KdV-type equations on the supercircle which are completely integrable for some values of the parameter only. We can only interpret some particular of these equations.

**0.1. Kirillov’s interpretation of the Schrödinger and Korteweg–de Vries operators.** Let  $\mathfrak{g} = \mathfrak{witt} = \mathfrak{der}\mathbb{C}[x^{-1}, x]$ ; let  $\hat{\mathfrak{g}} = \mathfrak{vir}$  be the nontrivial central extension of  $\mathfrak{g}$  given by the cocycle

$$\left[f \frac{d}{dx} + az, g \frac{d}{dx} + bz\right] = (fg' - f'g) \frac{d}{dx} + c \cdot \text{Res} f g''' \cdot z \text{ for } c \in \mathbb{C},$$

where  $z$  is the generator of the center of  $\hat{\mathfrak{g}}$ . Let  $\mathcal{F} = \mathbb{C}[x^{-1}, x]$  be the algebra of functions; let  $\mathcal{F}_\lambda$  be the rank 1 module over  $\mathfrak{g}$  and  $\mathcal{F}$  spanned over  $\mathcal{F}$  by  $dx^\lambda$ , where the  $\lambda$ th power of  $dx$  is determined via analyticity of the formula for the  $\mathfrak{g}$ -action:

$$\left(f \frac{d}{dx}\right)(dx^\lambda) = \lambda f' dx^\lambda.$$

In particular,  $\mathfrak{g} \cong \mathcal{F}_{-1}$ , as  $\mathfrak{g}$ -modules. Since the module  $\text{Vol}$  of volume forms is  $\mathcal{F}_1$ , the module dual to  $\mathfrak{g}$  is  $\mathfrak{g}^* = \mathcal{F}_2$ : we use one  $dx$  to kill  $\frac{d}{dx}$  and another  $dx$  to integrate the product of functions. (We confine ourselves to *regular* generalized functions, i.e., we ignore the elements from the space of functionals on  $\mathfrak{g}$  with 0-dimensional support, see [Kil].) Explicitly,

$$F(dx)^2 \left(f \frac{d}{dx}\right) = \text{Res } Ff.$$

- The Lie algebra of the stationary group of the element  $\hat{F} = (F, c) \in \hat{\mathfrak{g}}^* = (\mathfrak{g}^*, \mathbb{R} \cdot z^*)$  is

$$\mathfrak{st}_{\hat{F}} = \{\hat{X} \in \hat{\mathfrak{g}} : \hat{F}([\hat{X}, \hat{Y}]) = 0 \text{ for any } \hat{Y} \in \hat{\mathfrak{g}}\}.$$

Take  $\hat{X} = g \frac{d}{dx} + az$ ,  $\hat{Y} = f \frac{d}{dx} + bz$ . Then

$$\begin{aligned} \hat{F}([\hat{X}, \hat{Y}]) &= \hat{F}[(fg' - f'g) \frac{d}{dx} + \text{Res} f g''' \cdot z] = \\ &= \text{Res}[F(fg' - f'g) + cf g'''] \stackrel{(\text{partial integration})}{=} \text{Res}[Fg' + (Fg)' + cg''']. \end{aligned}$$

Hence,  $\hat{X} \in \mathfrak{st}_{\hat{F}}$  if and only if  $g$  is a solution of the equation  $L_3 g = 0$ . If  $c \neq 0$  we can always rescale the equation and assume that

$$c = 1. \tag{0.1.1}$$

In what follows this is understood. We call  $L_3$  the *KdV operator*. We also see that the KdV operator can be expressed in the form

$$L_3 = (\text{the cocycle operator that determines } \hat{\mathfrak{g}}) + \frac{d}{dx} F + F \frac{d}{dx}. \tag{0.1.2}$$

• Observe that the transformation properties of  $z^*$  and  $(dx)^2 \frac{d^2}{dx^2}$  under  $\mathfrak{g}$  are identical: scalars. The Schrödinger operator  $L_2 = \frac{d^2}{dx^2} + F$  is, clearly, selfadjoint. The factorization

$$F(dx)^2 + a \cdot z^* = (dx)^2 \left( F + a \frac{d^2}{dx^2} \right) \quad (0.1.3)$$

suggests to represent the elements of  $\hat{\mathfrak{g}}^*$  as 2nd order selfadjoint differential operators:  $\mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\lambda+2}$ . The selfadjointness fixes  $\lambda$ ; indeed,  $1 - (\lambda + 2) = \lambda$ , i.e.,  $\lambda = -\frac{1}{2}$ .

• Assume that  $F$  depends on time,  $t$  is the corresponding parameter. The *KdV hierarchy* is the series of evolution equations for  $L = L_2$  or, equivalently, for  $F$  :

$$\dot{L} = [L, A_k], \text{ where } A_k = (\sqrt{L}^{2k-1})_+ \text{ for } k = 1, 3, 5, \dots \quad (0.1.4 : \text{KdV}_k \text{ hierarchy})$$

Here the subscript  $+$  singles out the differential part of the pseudodifferential operator. The case  $k = 1$  is trivial and  $k = 3$  corresponds to the original KdV equation.

• Khesin and Malikov ([KM]) observed that we can also consider evolution equations for PDO:

$$\dot{L} = [L, A_\lambda], \text{ where } A_\lambda = \left( \sqrt{L} \right)^\lambda \text{ for } \lambda \in \mathbb{C}. \quad (0.1.5)$$

Such an approach to evolution equations for  $L_2$  is, as we will see, even more natural in the supersetting, when the Schrödinger operator becomes a pseudodifferential one itself.

**0.2. Kirillov's interpretation of supersymmetry of the Schrödinger and Korteweg–de Vries operators.** (To better understand this subsection, the reader has to know the technique of  $C$ -points, see [L1], [L2] and §1.) If in the above scheme we replace  $\mathfrak{g} = \mathfrak{m}|\mathfrak{t}$  with the Lie superalgebra  $\mathfrak{g} = \mathfrak{k}^L(1|1)$  of contact vector fields on the  $1|1$ -dimensional supercircle associated with the trivial bundle, we get two operators (for the definition of  $K_f$  see sec. 1.4)

$$\mathcal{L}_5 = K_\theta K_1^2 + 2FK_1 + 2K_1F + (-1)^{p(F)} K_\theta F K_\theta \quad (0.2.1 : \text{the } \mathfrak{ns} \text{ analog of } L_3, \text{ the KdV operator})$$

and

$$\mathcal{L}_3 = K_\theta K_1 + F. \quad (0.2.2 : \text{the } \mathfrak{ns} \text{ analog of } L_2, \text{ the Schrödinger operator})$$

Here  $F \in \Pi(\mathbb{C}[x^{-1}, x, \theta])$  and  $K_f$  is the contact vector field generated by  $f \in \mathbb{C}[x^{-1}, x, \theta]$ .

Indeed, let us calculate the stabilizer of an element of  $\mathfrak{ns}(1)^*$ . In doing so we will use the  $C$ -points of all objects encountered (physicists call such objects *superfields*).

Observe that since the integral (or residue) pairs 1 with  $\frac{\theta}{x}$ , this pairing is odd, and, therefore,  $\mathfrak{ns}(1)^* = \Pi(\mathcal{F}_3)$ .

The straightforward calculations yield:  $X = K_f \in \mathfrak{st}_{\mathcal{F}}$  if and only if  $f$  is a solution of the equation

$$\left( cK_\theta \frac{d^2}{dx^2} + 2\frac{d}{dx}F + 2F\frac{d}{dx} + (-1)^{p(F)} K_\theta F K_\theta \right) f = 0. \quad (0.2.3) : \text{KdV}(\mathfrak{ns}(1))$$

The operator

$$\mathcal{L}_5 = (\text{the cocycle operator that determines } \hat{\mathfrak{g}}) + 2FK_1 + 2K_1F + (-1)^{p(F)} K_\theta K_1 K_\theta \quad (0.2.4)$$

from the lhs of (0.2.3) will be called the  $\mathfrak{ns}(1)$ -KdV operator.

In components we have:  $f = f_0 + f_1\theta$ ,  $F = F_0 + F_1\theta$ , where  $f_0$  and  $F_1$  are even functions (of  $t$  with values in an auxiliary supercommutative superalgebra  $C$ ) while  $f_1$  and  $F_0$  are odd ones. Equation (KdV( $\mathfrak{ns}(1)$ )) takes the form:

$$\left[ \begin{pmatrix} L_3 & 0 \\ 0 & L_2 \end{pmatrix} + \begin{pmatrix} 0 & 2F_0\frac{d}{dx} + \frac{d}{dx}F_0 \\ F_0\frac{d}{dx} + 2\frac{d}{dx}F_0 & 0 \end{pmatrix} \right] \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = 0. \quad (0.2.5) : \begin{pmatrix} \text{KdV}(\mathfrak{ns}(1)) \\ \text{in components} \end{pmatrix}$$

Suppose  $F_0 = 0$ . Since formula (1.4.5) implies that  $\{f(t)\theta, g(t)\theta\}_{K.b.} = fg$ , we see that the product of two solutions of the Schrödinger equation  $L_2f = 0$  is a solution of the KdV equation  $L_3X = 0$ . This is Kirillov's supersymmetry.

*Remark .* Kirillov only considered  $\mathbb{R}$ -points of  $\mathfrak{g}$  and  $\hat{\mathfrak{g}}$ , that is why he missed all odd parameters of the supersymmetry he found — the second summand of the KdV operator

$$\mathcal{L}_5 = \begin{pmatrix} L_3 & 0 \\ 0 & L_2 \end{pmatrix} + \begin{pmatrix} 0 & 2F_0 \frac{d}{dx} + \frac{d}{dx} F_0 \\ F_0 \frac{d}{dx} + 2\frac{d}{dx} F_0 & 0 \end{pmatrix}. \quad (0.2.6)$$

**0.3. Our result.** We extend Kirillov's result from **witt** to all simple distinguished stringy Lie superalgebras. We thus elaborate the remark from [L1], p. 167, where the importance of odd parameters in this problem was first observed and the problem solved here was raised. To consider *all* superized KdV and Schrödinger operators was impossible before the list of stringy superalgebras and their cocycles (see [GLS]) was completed.

Passing to superization of the bulleted steps of sec. 0.1, we have to consider the elements of  $\hat{\mathfrak{g}}^*$  for the distinguished stringy superalgebras  $\mathfrak{g}$  as selfadjoint operators, perhaps, pseudodifferential, rather than differential. This together with ideas applied by Khesin–Malikov to the usual Schrödinger operator requires generalizations of the Lie superalgebra of matrices of complex size associated with the analogs of superprincipal embeddings of  $\mathfrak{osp}(N|2)$  for  $N \leq 4$ . Such superizations were recently described together with a description of the corresponding  $W$ -superalgebras and Gelfand–Dickey superalgebras, see [GL].

There remains a puzzle. In physical papers ([BIK], [DI], [DIK] and refs. therein) *parametric* families of KdV-type equations on the supercircle are introduced. The equations are completely integrable for some values of the parameter only. We can only identify some particular of these equations. How to describe the equations for the other values of parameter in terms of the supersymmetry algebra?

## §1. DISTINGUISHED STRINGY SUPERALGEBRAS

We recall all the necessary data. For the details of classification of simple finite dimensional Lie superalgebras see [K1], [LSch] and [GLS]; for a review of the representation theory of simple Lie superalgebras including infinite dimensional ones see [L2], for basics on supermanifolds see [L1] or [M]. The ground field is  $\mathbb{C}$ .

**1.1. Supercircle.** A *supercircle* or (for a physicist) a *superstring* of dimension  $1|n$  is the real supermanifold  $S^{1|n}$  associated with the rank  $n$  trivial vector bundle over the circle. Let  $x = e^{i\varphi}$ , where  $\varphi$  is the angle parameter on the circle, be the even indeterminate of the Fourier transforms; let  $\theta = (\theta_1, \dots, \theta_n)$ , be the odd coordinates on the supercircle formed by a basis of the fiber of the trivial bundle over the circle. Then  $(x, \theta)$  are the coordinates on  $(\mathbb{C}^*)^{1|n}$ , the complexification of  $S^{1|n}$ .

Denote by  $vol = vol(x, \theta)$  the volume element on  $(\mathbb{C}^*)^{1|n}$ . (Roughly speaking,  $vol$  “ = ”  $\frac{dx}{d\theta_1 \dots d\theta_n}$ , or even better  $vol$  “ = ”  $du_1 \cdot \dots \cdot du_m \cdot \frac{\partial}{\partial \xi_1} \cdot \dots \cdot \frac{\partial}{\partial \xi_n}$ . Recall that, actually, these are not equalities: as shown in [BL], the change of variables acts differently on the lhs and rhs and only coincides for the simplest transformations.)

Let the contact form be

$$\alpha = dx - \sum_{1 \leq i \leq n} \theta_i d\theta_i.$$

Usually, if  $\lfloor \frac{n}{2} \rfloor = k$  we rename the first  $2k$  indeterminates and express  $\alpha$  as follows for  $n = 2k$  and  $n = 2k + 1$ , respectively:

$$\alpha' = dx - \sum_{1 \leq i \leq k} (\xi_i d\eta_i + \eta_i d\xi_i) \text{ or } \alpha' = dx - \sum_{1 \leq i \leq k} (\xi_i d\eta_i + \eta_i d\xi_i) - \zeta d\zeta.$$

On  $(\mathbb{C}^*)^{1|n}$ , there are 4 series of simple “stringy” Lie superalgebras of vector fields and 4 exceptional such superalgebras. The 13 of them are distinguished: they admit nontrivial central extensions.

The “main” 3 series are:  $\mathbf{vect}^L(1|n) = \mathbf{der}\mathbb{C}[x^{-1}, x, \theta]$ , of all vector fields, its subalgebra  $\mathbf{svect}_\lambda^L(1|n)$  of vector fields that preserve the volume form  $x^\lambda \text{vol}$ , and  $\mathfrak{k}^L(1|n)$  that preserves the Pfaff equation  $\alpha = 0$ . The superscript  $L$  indicates that we consider vector fields with Laurent coefficients, not polynomial ones.

The Lie superalgebras of these 3 series are simple with the exception of  $\mathbf{svect}_\lambda^L(1|1)$  for any  $\lambda$ ,  $\mathbf{svect}_0^L(1|n)$  for  $n > 1$  and  $\mathfrak{k}^L(1|4)$ .

It so happens that  $\mathbf{svect}_0^L(1|n)$  contains a simple ideal of codimension  $\varepsilon^n$ , the quotient being spanned by  $\theta_1 \dots \theta_n \partial_x$ . Denote this ideal by  $\mathbf{svect}^{oL}(1|n)$ ; this is the fourth series.

The *twisted supercircle* of dimension  $1|n$  is the supermanifold that we denote  $S^{1|n-1;M}$  is associated with the Whitney sum of the trivial vector bundle of rank  $n - 1$  and the Moebius bundle. Since the Whitney sum of the two Moebius bundles is isomorphic to the trivial rank 2 bundle, we will only consider either  $S^{1|n}$  or  $S^{1|n-1;M}$ .

Let  $\theta_n^+ = \sqrt{x}\theta_n$  be the corresponding to the Moebius bundle odd coordinate on  $\mathbb{C}S^{1|n-1;M}$ , the complexification of  $S^{1|n-1;M}$ . Set

$$\tilde{\alpha} = dx - \sum_{1 \leq i \leq n-1} \theta_i d\theta_i - x\theta_n d\theta_n;$$

$$\tilde{\alpha}' = dx - \sum_{1 \leq i \leq k} (\xi_i d\eta_i + \eta_i d\xi_i) - x\theta_n d\theta_n; \text{ or } \tilde{\alpha}' = dx - \sum_{1 \leq i \leq k} (\xi_i d\eta_i + \eta_i d\xi_i) - \zeta d\zeta - x\theta_n d\theta_n.$$

and define the fifth series as the Lie superalgebra  $\mathfrak{k}^M(n)$  that preserves the Pfaff equation  $\tilde{\alpha} = 0$ .

One exceptional superalgebra,  $\mathfrak{m}^L(1)$ , is the Lie subsuperalgebra in  $\mathbf{vect}^L(1|1)$  that preserves the contact form

$$\beta = d\tau + \pi dq - q d\pi$$

corresponding to the “odd mechanics” on  $1|2$ -dimensional supermanifold. Though the following regradings demonstrate the isomorphism of this superalgebra with the nonexceptional ones, *considered as abstract* super algebras, they are distinct as filtered superalgebras and to various realizations of these Lie superalgebras different Schrödinger and KdV operators correspond.

Let  $t, \xi$  be the indeterminates for  $\mathbf{vect}(1|1)$ ; let  $t, \xi, \eta$  be same for  $\mathfrak{k}(1|2)$  (in the realization that preserves the Pfaff eq.  $\alpha' = 0$ ); and let  $\tau, q, \xi$  be the indeterminates for  $\mathfrak{m}(1)$ . Denote  $\mathbf{vect}(t, \xi)$  with the grading  $\deg t = 2$ ,  $\deg \xi = 1$  by  $\mathbf{vect}(t, \xi; 2, 1)$ , etc. Then the following exceptional nonstandard degrees indicated after a semicolon provide us with the isomorphisms:

$$\begin{aligned} \mathbf{vect}(t, \xi; 2, 1) &\cong \mathfrak{k}(1|2); & \mathfrak{k}(t, \xi, \eta; 1, 2, -1) &\cong \mathfrak{m}(1); \\ \mathbf{vect}(t, \xi; 1, -1) &\cong \mathfrak{m}(1); & \mathfrak{m}(\tau, q, \xi; 1, 2, -1) &\cong \mathfrak{k}(1|2). \end{aligned}$$

Another exception is the Lie superalgebra  $\mathfrak{k}^{Lo}(1|4)$ , the simple ideal of codimension 1 in  $\mathfrak{k}^L(1|4)$ , the quotient being generated by  $\frac{\theta_1 \theta_2 \theta_3 \theta_4}{x}$ . The remaining exceptions are not distinguished, so we ignore them in this paper.

**1.2. The modules of tensor fields.** To advance further, we have to recall the definition of the modules of tensor fields over the general vectoral Lie superalgebra  $\mathbf{vect}(m|n)$  and its subalgebras, see [BL]. Let  $\mathfrak{g} = \mathbf{vect}(m|n)$  (for any other  $\mathbb{Z}$ -graded vectoral Lie superalgebra the construction is identical) and  $\mathfrak{g}_{\geq} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ . Clearly,  $\mathbf{vect}_0(m|n) \cong \mathfrak{gl}(m|n)$ . Let  $V$  be the  $\mathfrak{gl}(m|n)$ -module with the *lowest* weight  $\lambda = \text{lwt}(V)$ . Make  $V$  into a  $\mathfrak{g}_{\geq}$ -module setting  $\mathfrak{g}_+ \cdot V = 0$  for  $\mathfrak{g}_+ = \bigoplus_{i > 0} \mathfrak{g}_i$ . Let us realize  $\mathfrak{g}$  by vector fields on the  $m|n$ -dimensional linear supermanifold  $\mathcal{C}^{m|n}$  with coordinates  $x = (u, \xi)$ . The superspace  $T(V) = \text{Hom}_{U(\mathfrak{g}_{\geq})}(U(\mathfrak{g}), V)$  is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to  $\mathbb{C}[[x]] \otimes V$ . Its elements have a natural interpretation as formal *tensor fields of type  $V$*  (or  $\lambda$ ). When  $\lambda = (a, \dots, a)$  we will simply write  $T(\vec{a})$  instead of  $T(\lambda)$ . We usually consider irreducible  $\mathfrak{g}_0$ -modules.

*Examples:*  $T(\vec{0})$  is the superspace of functions;  $\text{Vol}(m|n) = T(1, \dots, 1; -1, \dots, -1)$  (the semicolon separates the first  $m$  coordinates of the weight with respect to the matrix units  $E_{ii}$  of  $\mathfrak{gl}(m|n)$ ) is the superspace of *densities* or *volume forms*. We denote the generator of  $\text{Vol}(m|n)$  corresponding to the ordered set of coordinates  $x$  by  $\text{vol}(x)$ . The space of  $\lambda$ -densities is  $\text{Vol}^\lambda(m|n) = T(\lambda, \dots, \lambda; -\lambda, \dots, -\lambda)$ . In particular,  $\text{Vol}^\lambda(m|0) = T(\vec{\lambda})$  but  $\text{Vol}^\lambda(0|n) = T(-\vec{\lambda})$ .

**1.3. Modules of tensor fields over stringy superalgebras.** Denote by  $T^L(V) = \mathbb{C}[t^{-1}, t] \otimes V$  the  $\mathbf{vect}(1|n)$ -module that differs from  $T(V)$  by allowing the Laurent polynomials as coefficients of its elements instead of just polynomials. Clearly,  $T^L(V)$  is a  $\mathbf{vect}^L(1|n)$ -module. Define the *twisted with weight  $\mu$*  version of  $T^L(V)$  by setting:

$$T_\mu^L(V) = \mathbb{C}[t^{-1}, t]t^\mu \otimes V. \quad (1.3.1)$$

• **The “simplest” modules — the analogues of the standard or identity representation of the matrix algebras.** The simplest modules over the Lie superalgebras of series  $\mathbf{vect}$  are, clearly, the modules of  $\lambda$ -densities,  $\text{Vol}^\lambda$ . These modules are characterized by the fact that they are of rank 1 over  $\mathcal{F}$ , the algebra of functions. Over stringy superalgebras, we can also twist these modules and consider  $\text{Vol}_\mu^\lambda$ . Observe that for  $\mu \notin \mathbb{Z}$  this module has only one submodule, the image of the exterior differential  $d$ , see [BL], whereas for  $\mu \in \mathbb{Z}$  there is, additionally, the kernel of the residue:

$$\text{Res} : \text{Vol}^L \longrightarrow \mathbb{C}, \quad f \text{vol}_{t,\xi} \mapsto \text{the coefficient of } \frac{\xi_1 \cdots \xi_n}{t} \text{ in the expansion of } f. \quad (1.3.2)$$

• Over  $\mathbf{svect}^L(1|n)$  all the spaces  $\text{Vol}^\lambda$  are, clearly, isomorphic, since their generator,  $\text{vol}(t, \theta)$ , is preserved. So all rank 1 modules over the module of functions are isomorphic to the module of twisted functions  $\mathcal{F}_\mu$ .

Over  $\mathbf{svect}_\lambda^L(1|n)$ , the simplest modules are generated by  $t^\lambda \text{vol}(t, \theta)$ . The submodules of the simplest modules over  $\mathbf{svect}^L(1|n)$  and  $\mathbf{svect}_\lambda^L(1|n)$  are the same as those over  $\mathbf{vect}^L(1|n)$  but if  $\mu \in \mathbb{Z}$  there is, additionally, the trivial submodule generated by (the  $\lambda$ -th power of)  $\text{vol}(t, \theta)$  or  $t^\lambda \text{vol}(t, \theta)$ , respectively

• Over contact superalgebras  $\mathfrak{k}(2n+1|m)$ , it is more natural to express the simplest modules not in terms of  $\lambda$ -densities but via powers of the form  $\alpha$  which in what follows replaces  $\alpha'$  for the  $\mathfrak{k}^L$  series, or  $\tilde{\alpha}$  for the  $\mathfrak{k}^M$  series, or  $\beta$  for  $\mathfrak{m}^L(1)$ :

$$\mathcal{F}_\lambda = \begin{cases} \mathcal{F}\alpha^\lambda & \text{for } n = m = 0 \\ \mathcal{F}\alpha^{\lambda/2} & \text{otherwise} \end{cases}. \quad (1.3.3)$$

Observe that  $\text{Vol}^\lambda \cong \mathcal{F}_{\lambda(2n+2-m)}$  as  $\mathfrak{k}(2n+1|m)$ -modules. In particular, the Lie superalgebra of series  $\mathfrak{k}$  does not distinguish between  $\frac{\partial}{\partial t}$  and  $\alpha^{-1}$ : their transformation rules are identical.

Hence,

$$\mathfrak{k}(2n+1|m) \cong \begin{cases} \mathcal{F}_{-1} & \text{if } n = m = 0 \\ \mathcal{F}_{-2} & \text{otherwise} \end{cases}.$$

• For  $n = 0, m = 2$  (we take  $\alpha = dt - \xi d\eta - \eta d\xi$ ) there are other rank 1 modules over  $\mathcal{F}$ , the algebra of functions, namely:

$$T(\lambda, \nu)_\mu = \mathcal{F}_{\lambda;\mu} \cdot \left( \frac{d\xi}{d\eta} \right)^{\nu/2}. \quad (1.3.4)$$

• Over  $\mathfrak{k}^M$ , we should replace the form  $\alpha$  with  $\tilde{\alpha}$  and the definition of the  $\mathfrak{k}^L(1|m)$ -modules  $\mathcal{F}_{\lambda;\mu}$  should be replaced with

$$\mathcal{F}_{\lambda;\mu}^M = \begin{cases} \mathcal{F}_{\lambda;\mu}(\tilde{\alpha})^\lambda & \text{for } m = 1 \\ \mathcal{F}_{\lambda;\mu}(\tilde{\alpha})^{\lambda/2} & \text{for } m > 1. \end{cases} \quad (1.3.5)$$

• For  $m = 3$  and  $\alpha = dt - \xi d\eta - \eta d\xi - t\theta d\theta$  there are other rank 1 modules over the algebra of functions  $\mathcal{F}$ , namely:

$$T^M(\lambda, \nu)_\mu = \mathcal{F}_{\lambda;\mu}^M \cdot \left( \frac{d\xi}{d\eta} \right)^{\nu/2}. \quad (1.3.6)$$

*Examples .* 1) The  $\mathfrak{k}(2n+1|m)$ -module of volume forms is  $\mathcal{F}_{2n+2-m}$ . In particular,  $\mathfrak{k}(1|2) \subset \mathfrak{vect}(1|2)$ .

2) As  $\mathfrak{k}^L(1|m)$ -module,  $\mathfrak{k}^L(1|m)$  is isomorphic to  $\mathcal{F}_{-1}$  for  $m = 0$  and  $\mathcal{F}_{-2}$  otherwise. As  $\mathfrak{k}^M(1|m)$ -module,  $\mathfrak{k}^M(1|m)$  is isomorphic to  $\mathcal{F}_{-1}$  for  $m = 1$  and  $\mathcal{F}_{-2}$  otherwise. In particular,  $\mathfrak{k}^L(1|4) \simeq \text{Vol}$  and  $\mathfrak{k}^M(1|5) \simeq \Pi(\text{Vol})$ .

**1.4. Convenient formulas.** The four series of classical stringy superalgebras are  $\mathfrak{vect}^L(1|n)$ ,  $\mathfrak{vect}_\lambda^L(1|n)$ ,  $\mathfrak{k}^L(1|n)$  and  $\mathfrak{k}^M(1|n)$ .

$$D = f\partial_t + \sum f_i\partial_i \in \mathfrak{vect}_\lambda^L(1|n) \quad \text{if and only if} \quad \lambda f = -t\text{div}D. \quad (1.4.1)$$

A laconic way to describe  $\mathfrak{k}$ ,  $\mathfrak{m}$  and their subalgebras is via *generating functions*.

• Odd form  $\alpha = \alpha_1$ . For  $f \in \mathbb{C}[t, \theta]$  set :

$$K_f = \Delta(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,$$

where  $E = \sum_i \theta_i \frac{\partial}{\partial \theta_i}$ ,  $\Delta(f) = 2f - E(f)$ , and  $H_f$  is the hamiltonian field with Hamiltonian  $f$  that preserves  $d\alpha_1$ :

$$H_f = -(-1)^{p(f)} \left( \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_j} \right), \quad f \in \mathbb{C}[\theta].$$

The choice of the form  $\alpha'$  instead of  $\alpha$  only affects the form of  $H_f$  that we give for  $m = 2k + 1$ :

$$H_f = -(-1)^{p(f)} \sum_{j \leq k} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right), \quad f \in \mathbb{C}[\xi, \eta, \theta].$$

• Even form  $\beta = \alpha_0$ . For  $f \in \mathbb{C}[q, \xi, \tau]$  set:

$$M_f = \Delta(f) \frac{\partial}{\partial \tau} - L_e f - (-1)^{p(f)} \frac{\partial f}{\partial \tau} E,$$

where  $E = \sum_i y_i \frac{\partial}{\partial y_i}$  (here the  $y$  are all the coordinates except  $\tau$ ) is the Euler operator,  $\Delta(f) = 2f - E(f)$ , and

$$Le_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial q_i} \right), \quad f \in \mathbb{C}[q, \xi].$$

Since

$$L_{K_f}(\alpha_1) = K_1(f)\alpha_1, \quad L_{M_f}(\alpha_0) = -(-1)^{p(f)} M_1(f)\alpha_0, \quad (1.4.2)$$

it follows that  $K_f \in \mathfrak{k}(2n+1|m)$  and  $M_f \in \mathfrak{m}(n)$ . Observe that

$$p(Le_f) = p(M_f) = p(f) + \bar{1}.$$

• To the (super)commutators  $[K_f, K_g]$  or  $[M_f, M_g]$  there correspond *contact brackets* of the generating functions:

$$[K_f, K_g] = K_{\{f, g\}_{k.b.}}; \quad [M_f, M_g] = M_{\{f, g\}_{m.b.}}$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on  $t$  (resp.  $\tau$ ). The *Poisson bracket*  $\{\cdot, \cdot\}_{P.b.}$  is given by the formula

$$\begin{aligned} \{f, g\}_{P.b.} &= -(-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j} \text{ or} \\ \{f, g\}_{P.b.} &= -(-1)^{p(f)} \left[ \sum_{j \leq m} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial g}{\partial \xi_j} \right) + \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right]. \end{aligned} \quad (1.4.3)$$

The *Buttin bracket*  $\{\cdot, \cdot\}_{B.b.}$  is given by the formula

$$\{f, g\}_{B.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial q_i} \right). \quad (1.4.4)$$

In terms of the Poisson and Buttin brackets, respectively, the contact brackets take the form

$$\{f, g\}_{k.b.} = \Delta(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \Delta(g) - \{f, g\}_{P.b.} \quad (1.4.5)$$

and, respectively,

$$\{f, g\}_{m.b.} = \Delta(f) \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} \Delta(g) - \{f, g\}_{B.b.} \quad (1.4.6)$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\mathfrak{k}(2n+1|m) \cong \text{Span}(K_f : f \in \mathbb{C}[t, p, q, \xi]); \quad \mathfrak{m}(n) \cong \text{Span}(M_f : f \in \mathbb{C}[\tau, q, \xi]).$$

It is not difficult to verify that  $\mathfrak{k}^M(1|n) = \text{Span}(\tilde{K}_f : f \in R^L(1|n))$ , where the Möbius contact field is given by the formula

$$\tilde{K}_f = \Delta(f)\mathcal{D} + \mathcal{D}(f)E + \tilde{H}_f, \quad (1.4.7)$$

in which  $\Delta = 2 - E$ , but where

$$E = \sum_{i \leq n} \theta_i \frac{\partial}{\partial \theta_i} + \frac{1}{2} \theta \frac{\partial}{\partial \theta} \quad \text{and} \quad \mathcal{D} = \frac{\partial}{\partial t} - \frac{\theta}{2t} \frac{\partial}{\partial \theta} = \frac{1}{2} \tilde{K}_1$$

and where

$$\tilde{H}_f = (-1)^{p(f)} \left( \sum \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \theta_i} + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right)$$

in the realization with form  $\tilde{\alpha}$ ; in the realization with form  $\tilde{\alpha}'$  for  $n = 2k$  and  $n = 2k + 1$  we have, respectively:

$$\begin{aligned}\tilde{H}_f &= (-1)^{p(f)} \left( \sum \left( \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial}{\partial \xi_i} \right) + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right), \\ \tilde{H}_f &= (-1)^{p(f)} \left( \sum \left( \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial}{\partial \xi_i} \right) + \frac{\partial f}{\partial \zeta} \frac{\partial}{\partial \zeta} + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right).\end{aligned}$$

The corresponding contact bracket of generating functions will be called the *Ramond bracket*; its explicit form is

$$\{f, g\}_{R.b.} = \Delta(f)\mathcal{D}(g) - \mathcal{D}(f)\Delta(g) - \{f, g\}_{MP.b.}, \quad (1.4.8)$$

where the *Möbius-Poisson bracket*  $\{\cdot, \cdot\}_{MP.b.}$  is

$$\{f, g\}_{MP.b.} = (-1)^{p(f)} \left( \sum \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial \theta_i} + \frac{1}{t} \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial \theta} \right) \quad \text{in the realization with form } \tilde{\alpha}. \quad (1.4.9)$$

Observe that

$$L_{K_f}(\alpha_1) = K_1(f)\alpha_1, \quad L_{\tilde{K}_f}(\tilde{\alpha}) = \tilde{K}_1(f)\tilde{\alpha}. \quad (1.4.10)$$

### 1.5. Distinguished stringy superalgebras.

**Theorem .** *The only nontrivial central extensions of the simple stringy Lie superalgebras are those given in the following table.*

The operator  $\nabla$  introduced in the second column of the table by the formula  $c : D_1, D_2 \mapsto \text{Res}(D_1, \nabla(D_2))$  for an appropriate pairing  $(\cdot, \cdot)$  will be referred to as the *cocycle operator*.

Let in this subsection and in sec. 2.1  $K_f$  be the common notation of both  $K_f$  and  $\tilde{K}_f$ , depending on whether we consider  $\mathfrak{k}^L$  or  $\mathfrak{k}^M$ , respectively. Let further  $\mathcal{K} = (2\theta \frac{\partial}{\partial \theta} - 1) \frac{\partial}{\partial x^2}$  and let  $c : K_f, K_g \mapsto \text{Res}(K_f, \nabla(K_g))$  or  $c : M_f, M_g \mapsto \text{Res}(M_f, \nabla(M_g))$  be the cocycle that determines the nontrivial central extension.

algebra	the cocycle $c$	The extended algebra
$\mathfrak{k}^L(1 0)$	$\text{Res} f K_1^3(g)$	Virasoro or <b>vir</b>
$\left. \begin{array}{l} \mathfrak{k}^L(1 1) \\ \mathfrak{k}^M(1 1) \end{array} \right\}$	$\text{Res} f K_\theta K_1^2(g)$	Neveu-Schwarz or <b>ns</b> Ramond or <b>r</b>
$\left. \begin{array}{l} \mathfrak{k}^L(1 2) \\ \mathfrak{k}^M(1 2) \end{array} \right\}$	$(-1)^{p(f)} \text{Res} f K_{\theta_1} K_{\theta_2} K_1(g)$	2-Neveu-Schwarz or <b>ns</b> (2) 2-Ramond or <b>r</b> (2)
$\mathfrak{m}^L(1)$	$M_f, M_g \mapsto \text{Res} f (M_\xi)^3(g)$	$\widehat{\mathfrak{m}^L(1)}$
$\left. \begin{array}{l} \mathfrak{k}^L(1 3) \\ \mathfrak{k}^M(1 3) \end{array} \right\}$	$\text{Res} f K_\xi K_\theta K_\eta(g)$	3-Neveu-Schwarz or <b>ns</b> (3) 3-Ramond or <b>r</b> (3)
$\left. \begin{array}{l} \mathfrak{k}^{L_o}(4) \\ \mathfrak{k}^M(1 4) \end{array} \right\}$	$(1) (-1)^{p(f)} \text{Res} f K_{\theta_1} K_{\theta_2} K_{\theta_3} K_{\theta_4} K_1^{-1}(g)$	4-Neveu-Schwarz = <b>ns</b> (4) 4-Ramond = <b>ns</b> (4)
$\mathfrak{k}^{L_o}(4)$	$(2) \text{Res} f (tK_{t^{-1}}(g))$ $(3) \text{Res} f K_1(g)$	4'-Neveu-Schwarz = <b>ns</b> (4') 4 <sup>0</sup> -Neveu-Schwarz = <b>ns</b> (4 <sup>0</sup> )

Observe that  $K_1^{-1}$  is only defined on  $\mathfrak{k}^{L_o}(4)$  but not on  $\mathfrak{k}^L(4)$ ; though functions (2)  $\text{Res} f (tK_{t^{-1}}(g))$  and (3)  $\text{Res} f K_1(g)$  are defined on  $\mathfrak{k}^L(4)$  and  $\mathfrak{k}^M(4)$  they are not even cocycles there.

$\mathbf{vect}^L(1 2)$	the restriction of the cocycle (1) on $\mathfrak{k}^{L^\circ}(4)$ : $D_1 = f \frac{\partial}{\partial t} + g_1 \frac{\partial}{\partial \xi_1} + g_2 \frac{\partial}{\partial \xi_2}$ , $D_2 = \tilde{f} \frac{\partial}{\partial t} + \tilde{g}_1 \frac{\partial}{\partial \xi_1} + \tilde{g}_2 \frac{\partial}{\partial \xi_2}$ $\mapsto \text{Res}(g_1 \tilde{g}_2' - g_2 \tilde{g}_1' (-1)^{p(D_1)p(D_2)})$	$\widehat{\mathbf{vect}}^L(1 2)$
$\mathbf{svect}_\lambda^L(1 2)$	the restriction of the above	$\widehat{\mathbf{svect}}_\lambda^L(1 2)$
$\mathbf{vect}^L(1 1)$	$D_1 = f \frac{\partial}{\partial t} + g \frac{\partial}{\partial \xi}$ , $D_2 = \tilde{f} \frac{\partial}{\partial t} + \tilde{g} \frac{\partial}{\partial \xi} \mapsto$ $\text{Res}(f \mathcal{K}(\tilde{g}) + (-1)^{p(D_1)p(D_2)} g \mathcal{K}(\tilde{f}) +$ $2(-1)^{p(D_1)p(D_2)+p(D_2)} g \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta}(\tilde{g}))$	$\widehat{\mathbf{vect}}^L(1 1)$

Observe that the restriction of the only cocycle on  $\mathbf{vect}^L(1|2)$  to its subalgebra  $\text{Span}(f(t) \frac{\partial}{\partial t}) \cong \mathbf{witt}$  is trivial while the restriction of the only cocycle on  $\mathbf{svect}_\lambda^L(1|2)$  to its unique subalgebra  $\mathbf{witt}$  is nontrivial. The riddle is solved by a closer study of the embedding  $\mathbf{vect}(1|m) \longrightarrow \mathfrak{k}(1|2m)$ : it involves differentiations.

Explicitly, the embedding  $i : \mathbf{vect}^L(1|n) \longrightarrow \mathfrak{k}^L(1|2n)$  is given by the following formula in which  $\Phi = \sum_{i \leq n} \xi_i \eta_i$  (if we are not keen to preserve the isomorphism but are only interested

in a subalgebra of  $\mathfrak{k}^L(1|2n)$  isomorphic to  $\mathbf{vect}^L(1|n)$ , then the coefficients  $\frac{(-1)^{p(f)}}{2^m}$  can be dropped to simplify life):

$D \in \mathbf{vect}^L(1 n)$	the generator of $i(D)$
$f(\xi) t^m \partial_t$	$(-1)^{p(f)} \frac{1}{2^m} f(\xi) (t - \Phi)^m$
$f(\xi) t^m \partial_i$	$(-1)^{p(f)} \frac{1}{2^m} f(\xi) \eta_i (t - \Phi)^m$

(1.5.1)

Clearly,  $\mathbf{svect}_\lambda^L(1|n)$  is the subsuperspace of  $\mathbf{vect}^L(1|n)$  spanned by the expressions

$$f(\xi)(t - \Phi)^m + \sum_i f_i(\xi) \eta_i (t - \Phi)^{m-1} \text{ such that } (\lambda + n)f(\xi) = - \sum_i (-1)^{p(f_i)} \frac{\partial f_i}{\partial \xi_i}. \quad (1.5.2)$$

The nonzero values of the cocycle  $c$  on  $\mathbf{vect}^L(1|2)$  in monomial basis are:

$$\begin{aligned} c(t^k \theta_1 \frac{\partial}{\partial \theta_1}, t^l \theta_2 \frac{\partial}{\partial \theta_2}) &= k \delta_{k,-l}, & c(t^k \theta_1 \frac{\partial}{\partial \theta_2}, t^l \theta_2 \frac{\partial}{\partial \theta_1}) &= -k \delta_{k,-l}, \\ c(t^k \theta_1 \theta_2 \frac{\partial}{\partial \theta_1}, t^l \frac{\partial}{\partial \theta_2}) &= k \delta_{k,-l}, & c(t^k \theta_1 \theta_2 \frac{\partial}{\partial \theta_2}, t^l \frac{\partial}{\partial \theta_1}) &= k \delta_{k,-l}. \end{aligned} \quad (1.5.3)$$

In  $\mathbf{svect}_\lambda^L(1|2)$ , set:

$$\begin{aligned} L_m &= t^m \left( t \frac{\partial}{\partial t} + \frac{\lambda + m + 1}{2} (\theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2}) \right), \\ S_m^j &= t^m \theta_j \left( t \frac{\partial}{\partial t} + (\lambda + m + 1) (\theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2}) \right). \end{aligned} \quad (1.5.4)$$

The nonzero values of the cocycles on  $\mathbf{svect}_\lambda^L(1|2)$  are

$$\begin{aligned} c(L_m, L_n) &= \frac{1}{2} m(m^2 - (\lambda + 1)^2) \delta_{m,-n}, \\ c(t^k \frac{\partial}{\partial \theta_i}, S_m^j) &= -m(m - (\lambda + 1)) \delta_{m,-n} \delta_{i,j}, \\ c(t^m (\theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2}), t^n (\theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2})) &= m \delta_{m,-n}, \\ c(t^m \theta_1 \frac{\partial}{\partial \theta_2}, t^n \theta_2 \frac{\partial}{\partial \theta_1}) &= m \delta_{m,-n}. \end{aligned} \quad (1.5.5)$$

## §2. SUPERIZED KdV AND SCHRÖDINGER OPERATORS

**2.1. The KdV operators for the distinguished contact superalgebras.** Let in this subsection  $K_f$  be the common term for both  $K_f$  and  $\tilde{K}_f$ , as Table 1.5, for  $\mathfrak{g} = \mathfrak{k}^L$  or  $\mathfrak{k}^{L^\circ}$  or  $\mathfrak{k}^M$ . The equation for  $K_f \in \mathfrak{st}_{(F,1)}$ , where  $(F, 1) \in \hat{\mathfrak{g}}^*$ , is of the form  $KdV(f) = 0$ , where the operators  $KdV$  are listed in the following table with the cocycle operators being the symmetrizations of the operators  $\nabla$  from sec. 1.5.

$n$	$KdV = \text{the cocycle operator} \oplus \text{“the standard part”}$
0	$\frac{d^3}{x^3} \oplus F \frac{d}{x} + \frac{d}{dx} F$
1	$K_\theta(K_1)^2 \oplus 2(F\partial_x + \partial_x F) + (-1)^{p(F)} K_\theta F K_\theta$
2	$(K_\xi K_\eta - K_\eta K_\xi) K_1 \oplus 2(F\partial_x + \partial_x F) + (-1)^{p(F)} (K_\xi F K_\eta - K_\eta F K_\xi)$
3	$(K_\xi K_\eta - K_\eta K_\xi) K_\theta \oplus 2F\partial_x + 2\partial_x F + (-1)^{p(F)} (K_\xi F K_\eta - K_\eta F K_\xi + K_\theta F K_\theta)$
$4_1$	$(K_{\xi_1} K_{\eta_1} - K_{\eta_1} K_{\xi_1})(K_{\xi_2} K_{\eta_2} - K_{\eta_2} K_{\xi_2}) \int_x \oplus 2(F\partial_x + \partial_x F) + (-1)^{p(F)} \sum_{i=1,2} (K_{\xi_i} F K_{\eta_i} - K_{\eta_i} F K_{\xi_i})$
$4_2$	$x K_{x^{-1}} \oplus \text{the standard part from } 4_1$
$4_3$	$K_1 \oplus \text{the standard part from } 4_1$

The KdV operator can, therefore, be always represented in the following form with “the standard part” explicified:

$$\text{the cocycle operator} + 2(F\partial_x + \partial_x F) + (-1)^{p(F)} \sum (K_{\xi_i} F K_{\eta_i} - K_{\eta_i} F K_{\xi_i}) \quad (\text{for } \mathfrak{k}^L(1|n))$$

$$\begin{aligned} & \text{the cocycle operator} + 2(F\partial_x + \partial_x F) - F \frac{\partial}{x} \partial_\theta - \frac{\partial}{x} \partial_\theta F + \frac{F(1-E)}{x} + \\ & (-1)^{p(F)} [\sum_{i \leq n-1} K_{\theta_i} F K_{\theta_i} + t K_\theta F K_\theta] + \frac{1}{2x} [\theta \partial_\theta F \sum_{i \leq n-1} \theta_i \partial_{\theta_i} - \sum_{i \leq n-1} \theta_i \partial_{\theta_i} \theta \partial_\theta]. \end{aligned}$$

(for  $\mathfrak{k}^M(1|n)$ )

So the KdV operators corresponding to the supercircles associated with the cylinder and the Möbius band are absolutely different. To establish that similar is the situation with the Schrödinger operators, let us compair  $\mathfrak{g}$  with  $\mathfrak{g}^*$  for the cylinder and the Möbius band:

$\mathfrak{g} = \mathfrak{k}^L(1 n)$	0	1	2	3	4	5	6	7
$\mathfrak{g}$	$\mathcal{F}_{-1}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$
$Vol$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	$\mathcal{F}_{-3}$	$\Pi(\mathcal{F}_{-4})$	$\mathcal{F}_{-5}$	$\Pi(\mathcal{F}_{-6})$
$\mathfrak{g}^*$	$\mathcal{F}_2$	$\Pi(\mathcal{F}_3)$	$\mathcal{F}_2$	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_0$	$\Pi(\mathcal{F}_{-1})$	$\mathcal{F}_{-2}$	$\Pi(\mathcal{F}_{-3})$

$\mathfrak{g} = \mathfrak{k}^M(1 n)$	1	2	3	4	5	6	7	$n$
$\mathfrak{g}$	$\mathcal{F}_{-1}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$
$Vol$	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	$\mathcal{F}_{-3}$	$\Pi(\mathcal{F}_{-4})$	$\Pi^n(\mathcal{F}_{3-n})$
$\mathfrak{g}^*$	$\Pi(\mathcal{F}_2)$	$\mathcal{F}_3$	$\Pi(\mathcal{F}_2)$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	$\Pi^n(\mathcal{F}_{5-n})$

The comparison of  $\mathfrak{g}$  with  $\mathfrak{g}^*$  shows that there is a nondegenerate bilinear form on  $\mathfrak{g} = \mathfrak{k}^L(1|6)$  and  $\mathfrak{g} = \mathfrak{k}^M(1|7)$ , even and odd, respectively. These forms are supersymmetric and given by the formula

$$(K_f, K_g) = \text{Res } fg.$$

**2.2. Examples.**  $1_{\mathfrak{ns}(1)}$  is considered in sec. 0.2.

$1_{\mathfrak{r}(1)} \hat{\mathbf{g}} = \mathfrak{r}(1)$  Let us calculate the stabilizer of an element of  $\mathfrak{r}(1)^*$ . In doing so we will use the  $C$ -points of all objects encountered. Observe again that since the integral (residue) pairs 1 with  $\frac{\theta}{t}$ , this pairing is odd, and, therefore,  $\mathfrak{r}(1)^* = \Pi(\mathcal{F}_2)$ .

In components the equation (for  $n = 1$ ) takes the form:

$$\left[ \begin{pmatrix} L_3 & 0 \\ 0 & L_2 \end{pmatrix} + \begin{pmatrix} 0 & 2F_0 \frac{d}{dx} + \frac{d}{dx} F_0 - \frac{F_0}{x} \\ F_0 \frac{d}{dx} + 2\frac{d}{dx} F_0 + \frac{F_0}{x} & 0 \end{pmatrix} \right] \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = 0,$$

where  $L_3 = \frac{d^3}{dx^3} + 2\frac{d}{dx} F_1 + 2F_1 \frac{d}{dx}$ ,  $L_2 = \frac{1}{x}(\frac{\partial}{\partial x} - \frac{1}{2x})^2 - \frac{F_1}{x}$ .

$2_{\mathfrak{ns}(2)}$  In components the analog of KdV corresponding to  $\mathfrak{ns}(2)$  is

$$\left[ \begin{pmatrix} L_3 & 2F_0 \frac{d}{dx} & 0 & 0 \\ 2F_0 \frac{d}{dx} & \frac{d}{dx} & 0 & 0 \\ 0 & 0 & 0 & L_2 \\ 0 & 0 & L_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -(2F_2 \frac{d}{dx} + \frac{d}{dx} F_2) & 2F_1 \frac{d}{dx} + \frac{d}{dx} F_1 \\ 0 & 0 & F_2 & -F_1 \\ -(F_2 \frac{d}{dx} + 2\frac{d}{dx} F_2) & -F_2 & 0 & 0 \\ F_1 \frac{d}{dx} + 2\frac{d}{dx} F_1 & F_1 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} f_0 \\ f_{12} \\ f_1 \\ f_2 \end{pmatrix} = 0,$$

where  $f = f_0 + f_1 \xi + f_2 \eta + f_{12} \xi \eta$ ,  $F = F_0 + F_1 \xi + F_2 \eta + F_{12} \xi \eta$  and

$$L_3 = \frac{d^3}{dx^3} + 2F_{12} \frac{d}{dx} + 2\frac{d}{dx} F_{12}; \quad L_2 = \frac{d^2}{dx^2} + F_{12} + F_0 \frac{d}{dx} + \frac{d}{dx} F_0.$$

**2.3. The list of Schrödinger operators.** In the following table we use an abbreviation:  $\Delta = \tilde{K}_\theta(\tilde{K}_1)^{-1} - (\tilde{K}_1)^{-1} \tilde{K}_\theta$ . The Schrödinger operator is the sum of the operator given in the tables with a potential  $F$ , where  $F \in \mathcal{F}$  or  $F \in \Pi(\mathcal{F})$ : the parity of the potential should be equal to that of the operator.

$n$	0	1	2	3
$\mathfrak{k}^L(1 n)$	$K_1^2$	$K_\theta K_1$	$K_\xi K_\eta - K_\eta K_\xi$	$(K_\xi K_\theta K_\eta - K_\eta K_\theta K_\xi)(K_1)^{-1}$
$\mathfrak{k}^M(1 n)$	—	$\Delta \tilde{K}_1^2$	$\Delta \tilde{K}_{\theta_1} \tilde{K}_1$	$\Delta(\tilde{K}_\xi \tilde{K}_\eta - \tilde{K}_\eta \tilde{K}_\xi)$
$\mathfrak{k}^L(1 4)$	(1) $(K_{\xi_1} K_{\eta_1} - K_{\eta_1} K_{\xi_1})(K_{\xi_2} K_{\eta_2} - K_{\eta_2} K_{\xi_2})(K_1)^{-2}$ (2) $x K_{x^{-1}} (K_1)^{-1} - (K_1)^{-1} x K_{x^{-1}}$ (3) ??			
$\mathfrak{k}^M(1 4)$	(1) $\Delta \tilde{K}_{\theta_1} (\tilde{K}_\xi \tilde{K}_\eta - \tilde{K}_\eta \tilde{K}_\xi)(\tilde{K}_1)^{-1}$			

So far we failed to write an explicit expression for the third Schrödinger operators.

For the Lie superalgebra  $\mathfrak{vect}^L(1|1)$  the Schrödinger operator is the same operator as for  $\mathfrak{k}^L(1|2)$  but rewritten in the form of a matrix and with  $\eta$  replaced with  $\partial_\xi$ . We leave as an exercise to the reader the pleasure to write this matrix explicitly as well as to reexpress it in terms of the fields  $M_f$  for  $\mathfrak{m}^L(1)$ .

For  $\mathfrak{vect}^L(1|2)$  the Schrödinger operators are obtained from the Schrödinger operator

$$\begin{pmatrix} \xi_1 \xi_2 \partial_{\xi_2} \partial_x^2 & \xi_2 \partial_{\xi_1} - \partial_{\xi_2} \partial_x & \xi_1 \partial_{\xi_1} \partial_{\xi_2} \partial_x & \partial_{\xi_1} \partial_{\xi_2} \\ -\xi_1 \xi_2 \partial_{\xi_2} \partial_x^3 & -\xi_1 \xi_2 \partial_{\xi_1} \partial_{\xi_2} \partial_x^2 & \xi_1 \partial_{\xi_2} \partial_x^2 & -\xi_1 \partial_{\xi_1} \partial_{\xi_2} \partial_x \\ \xi_1 \xi_2 \partial_{\xi_1} \partial_x^3 & \xi_1 \partial_{\xi_1} \partial_x^2 & \xi_2 \partial_{\xi_2} \partial_x^2 & \xi_2 \partial_{\xi_1} \partial_{\xi_2} \partial_x \\ -\xi_1 \xi_2 \partial_{\xi_1} \partial_x^4 & -\xi_1 \xi_2 \partial_{\xi_1} \partial_x^3 & -\xi_1 \xi_2 \partial_{\xi_2} \partial_x^3 & -\xi_1 \xi_2 \partial_{\xi_1} \partial_{\xi_2} \partial_x^2 \end{pmatrix} \partial_x^{-2} \quad (*)$$

for  $\mathfrak{k}^L(1|4)$  after restriction to the subalgebra  $\mathbf{vect}^L(1|2)$ , see (1.2). In the dual space this leads to the quotient space: we should disregard the first row and the last column of  $(*)$ .

**2.4. The Schrödinger operators as selfadjoint differential operators.** • For the Neveu–Schwarz superalgebras we have the exact sequences

$$0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{ns}(n) \longrightarrow \mathcal{F}_2 \longrightarrow 0. \quad (2.4.1)$$

Here  $\mathfrak{z} = \text{Span}(z)$  if  $n \neq 4$  and  $\mathfrak{z} = \text{Span}(z, z_1, z_2)$  if  $n = 4$ .

Using the identification  $\text{Vol} \cong \Pi^n(\mathcal{F}_{2-n})$  we dualize the above exact sequence and get:

$$\begin{aligned} 0 \longrightarrow \Pi^n(\mathcal{F}_{4-n}) \longrightarrow \mathfrak{ns}^*(n) \longrightarrow \mathfrak{z}^* \longrightarrow 0 & \text{ for } n = 1, 2, 3, \\ 0 \longrightarrow \mathcal{F}_0/\mathbb{C} \longrightarrow \mathfrak{ns}^*(4) \longrightarrow \mathfrak{z}^* \longrightarrow 0, \\ 0 \longrightarrow \mathcal{F}_0/\mathbb{C} \longrightarrow \mathfrak{ns}^*(4') \longrightarrow \mathfrak{z}^* \longrightarrow 0, \\ 0 \longrightarrow \mathcal{F}_0/\mathbb{C} \longrightarrow \mathfrak{ns}^*(4^0) \longrightarrow \mathfrak{z}^* \longrightarrow 0. \end{aligned} \quad (2.4.2)$$

For the cocycles (2) and (3) from Table 1.5 we can as well consider not the  $\mathfrak{ns}^*(4')$  and  $\mathfrak{ns}^*(4^0)$  but the algebras of their derivations; than we do not have to factorize modulo constants which simplifies life.

• For the Ramond superalgebras we similarly have the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{r}(1) \longrightarrow \mathcal{F}_{-1} \longrightarrow 0 \\ 0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{r}(n) \longrightarrow \mathcal{F}_{-2} \longrightarrow 0 \text{ for } n > 1. \end{aligned} \quad (2.4.3)$$

Here  $\mathfrak{z} = \text{Span}(z)$ .

Using the identification  $\text{Vol} \cong \begin{cases} \Pi(\mathcal{F}_1) & \text{for } n = 1 \\ \Pi^n(\mathcal{F}_{3-n}) & \text{for } n > 1 \end{cases}$  we dualize the above exact sequence and get:

$$\begin{aligned} 0 \longrightarrow \Pi(\mathcal{F}_2) \longrightarrow \mathfrak{r}^*(1) \longrightarrow \mathfrak{z}^* \longrightarrow 0 \\ 0 \longrightarrow \Pi^n(\mathcal{F}_{5-n}) \longrightarrow \mathfrak{r}^*(n) \longrightarrow \mathfrak{z}^* \longrightarrow 0. \end{aligned} \quad (2.4.4)$$

Let us realize the elements of  $\mathfrak{ns}^*(n)$  and  $\mathfrak{r}^*(n)$  by selfadjoint (pseudo)differential operators  $\hat{F} : \mathcal{F}_\lambda \longrightarrow \Pi^n(\mathcal{F}_\mu)$ . We have already done this for  $\mathbf{vir}$  in Introduction. The order of  $\hat{F}$  is equal to  $4 - n$  for  $\mathfrak{ns}^*(n)$ ; it is equal to  $5 - n$  for  $\mathfrak{r}^*(n)$  if  $n > 1$  and 2 for  $\mathfrak{r}^*(1)$ .

Now, let us solve the systems of two equations, of which the first equation counts the order of Sch and the second one is the dualization condition:

$$\begin{aligned} \mu = 2 + (2 - n) + \lambda, \quad \mu + \lambda = 2 - n & \text{ for } \mathfrak{ns}(n) \\ \mu = 1 + 1 + \lambda, \quad \mu + \lambda = 1 & \text{ for } \mathfrak{r}(1) \\ \mu = 2 + (3 - n) + \lambda, \quad \mu + \lambda = 3 - n & \text{ for } \mathfrak{r}(n), n > 1. \end{aligned}$$

The solutions are:

$$\begin{aligned} \mu = 3 - n, \quad \lambda = -1 & \text{ for } \mathfrak{ns}(n) \\ \mu = \frac{3}{2}, \quad \lambda = -\frac{1}{2} & \text{ for } \mathfrak{r}(1) \\ \mu = 4 - n, \quad \lambda = -1 & \text{ for } \mathfrak{r}(n), n > 1. \end{aligned}$$

**2.5. The KdV hierarchies associated with the Schrödinger operators.** Let  $L_r$  be the Schrödinger operator of order  $r$ , see sec. 2.3. Define the KdV-type equation as the following Lax  $L$ - $A$  pairs:

$$D_{\mathcal{T}}(L) = [L, A_k], \text{ where } A_k = (L^{k/r})_+ \text{ for } k \not\equiv r \pmod{r} \quad (2.5.1)$$

and where

$$D_{\mathcal{T}} = \begin{cases} \frac{d}{dx} & \text{if } p(A_k) = \bar{0} \\ \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial x} & \text{if } p(A_k) = \bar{1}. \end{cases}$$

Here the subscript  $+$  singles out the differential part of the pseudodifferential operator. For complex  $k$  the differential part is not well-defined and we shall proceed, mutatis mutandis, as Khesin–Malikov, namely, by setting

$$D_{\mathcal{T}}(L) = [L, A_k], \text{ where } A_k = (L^{1/r})^k \text{ for } k \in \mathbb{C}. \quad (2.5.2)$$

**Pseudodifferential operators on the supercircle.** Let  $V$  be a superspace. For  $\theta = (\theta_1, \dots, \theta_n)$  set

$$\begin{aligned} V[x, \theta] &= V \otimes \mathbb{K}[x, \theta]; \quad V[x^{-1}, x, \theta] = V \otimes \mathbb{K}[x^{-1}, x, \theta]; \\ V[[x^{-1}, \theta]] &= V \otimes \mathbb{K}[[x^{-1}, \theta]]; \\ V((x, \theta)) &= V \otimes \mathbb{K}[[x^{-1}]] [x, \theta]. \end{aligned}$$

We call  $V((x, \theta))x^\lambda$  the space of *pseudodifferential symbols*. Usually,  $V$  is a Lie (super)algebra. Such symbols correspond to pseudodifferential operators (pdo) of the form

$$\sum_{i=-\infty}^n \sum_{k_0+\dots+k_n=i} a_i(\partial_x)^{k_0} \theta_1^{k_1} \dots \theta_n^{k_n},$$

Here  $k_i = 0$  or  $1$  for  $i > 0$  and  $a_i(x, \theta) \in V$ . This is clear.

For any  $P = \sum_{i \leq m} P_i x^i \theta_0^k \theta^j \in V((x, \theta))$  we call  $P_+ = \sum_{i, j, k \geq 0} P_i x^i \theta_0^k \theta^j$  the *differential part* of  $P$  and  $P_- = \sum_{i, k < 0} P_i x^i \theta_0^k \theta^j$  the *integral part* of  $P$ .

The space  $\Psi DO$  of pdos is, clearly, the left module over the algebra  $\mathcal{F}$  of functions. Define the left  $\Psi DO$ -action on  $\mathcal{F}$  from the Leibniz formula thus making  $\Psi DO$  into a superalgebra.

Define the involution in the superalgebra  $\Psi DO$  setting

$$(a(t, \theta) D^i \tilde{D}^j)^* = (-1)^{jip(\tilde{D})p(D)} \tilde{D}^j D^i a^*(x, \theta).$$

The following fact is somewhat unexpected. If  $D$  is an odd differential operator, then  $D^2$  is well-defined as  $\frac{1}{2}[D, D]$ . Hence, we can consider the set  $V((x, \theta))x^\lambda$  for an *odd*  $x$ ! Therefore, there are two types of pdos: *contact* ones, when  $D^2 \neq 0$  for odd  $D$ 's and general ones, when all odd  $D$ 's are nilpotent.

*Conjecture* . There exists a residue for all distinguished dimensions, i.e. for the contact type pdos in dimensions  $1|n$  for  $n \leq 4$  and for the general pdos in dimensions  $1|n$  for  $n \leq 2$ .

So far, however, the residue was defined only for contact type pdos, of  $\mathfrak{k}^L$  type, and only for  $n = 1$  at that.

Extend [MR] and define the *residue* of  $P = \sum_{i \leq m} P_i x^i \theta_0^k \theta^j \in V((x, \theta_0, \theta))$  for  $n = 1$ . We can do it thanks to the following exceptional property of  $\mathfrak{k}^L(1|1)$  and  $\mathfrak{k}^M(1|1)$ . Indeed, over  $\mathfrak{k}^L(1|1)$ , the volume form “ $dt \frac{\partial}{\partial \theta}$ ” is, more or less,  $d\theta$ : consider the quotient  $\Omega^1/\mathcal{F}\alpha$ , where  $\alpha$  is the contact form preserved by  $\mathfrak{k}^L(1|1)$ ; similarly, over  $\mathfrak{k}^M(1|1)$ , the transformation rules of  $dt \frac{\partial}{\partial \theta}$  and  $\tilde{\alpha}$ , where  $\tilde{\alpha}$  is the contact form preserved by  $\mathfrak{k}^M(1|1)$ , are identical. Therefore, define the residue by the formula

$$\text{Res } P = \text{coefficient of } \frac{\theta}{x} \text{ in the expansion of } P_{-1}.$$

*Remark* . Manin and Radul [MR] considered the Kadomtsev–Petviashvili hierarchy associated with  $\mathfrak{ns}$ , i.e., for  $D = K_\theta$ . The formula for the residue allows one to directly generalize their result and construct a simialr hierarchy associated with  $\mathfrak{r}$ , i.e., for  $D = \tilde{K}_\theta$ .

This new phenomenon — an invertible odd symbol — doubles the old picture: let  $\theta_0$  be the symbol of  $D$ , and let  $x$  be the symbol of the differential operator  $D^2$ . We see that the case of the odd  $D$  reduces to either  $V((x, \theta_0, \theta))x^\lambda$  or  $V((x, \theta_0^{-1}, \theta))x^\lambda$ . This is in accordance with the fact that every irreducible finite dimensional representation of  $\mathfrak{osp}(1|2)$  is glued of two representations of  $\mathfrak{sl}(2)$ .

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